# Remarks on curvature dimension conditions on graphs

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**Abstract**

We show a connection between the $CDE'$ inequality introduced in [4] and the $CD\psi$ inequality established in [5]. In particular, we introduce a $CD_\psi^\varphi$ inequality as a slight generalization of $CD\psi$ which turns out to be equivalent to $CDE'$ with appropriate choices of $\varphi$ and $\psi$. We use this to prove that the $CDE'$ inequality implies the classical $CD$ inequality on graphs, and that the $CDE'$ inequality with curvature bound zero holds on Ricci-flat graphs.

## 1 Introduction

There is an immense interest in the heat equation on graphs. In this context, curvature-dimension conditions have attracted particular attention. In particular, recent works [2, 4, 5] have introduced a variety of such conditions. In this note, we will extend ideas of [5] to show a connection between them (Proposition 2.3 and Section 3). Moreover, we will prove that Ricci-flat graphs satisfy the $CDE'$ condition (Section 4).

Throughout the note, we will use notation and definitions introduced in [1, 2, 3, 4, 5] which can also be found in the appendix.

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## 2 The connection between the $CDE'$ and the $CD\psi$ inequality

First, we consider the connection between $\Gamma$ (cf. Definition A.6) and $\Gamma^\psi$ (cf. Definition A.11), and between $\widetilde{\Gamma}_2$ (cf. Definition A.8) and $\Gamma_2^\psi$ (cf. Definition A.12).

**Lemma 2.1.** *For all $f \in C^+(V)$,*

$$f\Gamma^{\sqrt{\cdot}}(f) = \Gamma(\sqrt{f}), \tag{2.1}$$

$$f\Gamma_2^{\sqrt{\cdot}}(f) = \widetilde{\Gamma}_2(\sqrt{f}). \tag{2.2}$$

*Proof.* Let $f \in C^+(V)$ and $x \in V$. Then for the proof of (2.1),

$$2\left[f\Gamma^{\sqrt{\cdot}}(f)\right](x) = 2f(x)\left[\frac{\Delta f}{2f} - \Delta\sqrt{\frac{f}{f(x)}}\right](x) = \left[\Delta f - 2\sqrt{f}\Delta\sqrt{f}\right](x) = 2\Gamma(\sqrt{f})(x).$$

Next, we prove (2.2). In [2, (4.7)], it is shown that for all positive solutions $u \in C^1(V \times \mathbb{R}_0^+)$ to the heat equation, one has

$$2\widetilde{\Gamma}_2 \sqrt{u} = \mathcal{L}(\Gamma \sqrt{u}).$$

Now, we set $u := P_t f$ and we apply the above proven identity (2.1) and the identity $2u\Gamma_2^{\psi}(u) = \mathcal{L}(u\Gamma^{\psi}(u))$ (cf. [5, Subsection 3.2]) to obtain

$$2\widetilde{\Gamma}_2(\sqrt{f}) = \left[\mathcal{L}(\Gamma(\sqrt{u}))\right]_{t=0} = \left[\mathcal{L}(u\Gamma^{\sqrt{\cdot}}(u))\right]_{t=0} = 2f\Gamma_2^{\sqrt{\cdot}}(f).$$

This finishes the proof. $\square$

The following definition extends the $CD\psi$ inequality to compare it to the $CDE'$ inequality.

**Definition 2.2** ($CD_{\psi}^{\varphi}$ condition)**.** Let $d \in (0, \infty]$ and $K \in \mathbb{R}$. Let $\varphi, \psi \in C^1(\mathbb{R}^+)$ be concave functions. A graph $G = (V, E)$ satisfies the $CD_{\psi}^{\varphi}(d, K)$ condition, if for all $f \in C^+(V)$,

$$\Gamma_2^{\psi}(f) \geq \frac{1}{d}\left(\Delta^{\varphi} f\right)^2 + K\Gamma^{\psi}(f).$$

Indeed, this definition is an extension of $CD\psi$ which is equivalent to $CD_{\psi}^{\psi}$.

**Proposition 2.3.** *Let $G = (V, E)$ be a graph, let $d \in (0, \infty]$ and $K \in \mathbb{R}$. Then, the following statements are equivalent.*

*(i) $G$ satisfies the $CDE'(d, K)$ inequality.*

*(ii) $G$ satisfies the $CD_{\sqrt{\cdot}}^{\log}(4d, K)$ inequality.*

*Proof.* By definition, the $CDE'(d, K)$ inequality is equivalent to

$$\widetilde{\Gamma}_2(f) \geq \frac{1}{d} f^2 \left(\Delta \log f\right)^2 + K\Gamma(f), \quad f \in C^+(V).$$

By replacing $f$ by $\sqrt{f}$ (all allowed $f \in C(V)$ are strictly positive), this is equivalent to

$$\widetilde{\Gamma}_2(\sqrt{f}) \geq \frac{1}{d} f \left(\Delta \log \sqrt{f}\right)^2 + K\Gamma(\sqrt{f}), \quad f \in C^+(V).$$

By applying Lemma 2.1 and the fact that $\Delta^{\log} = \Delta \circ \log$, this is equivalent to

$$f\Gamma_2^{\sqrt{\cdot}}(f) \geq \frac{1}{4d} f \left(\Delta^{\log} f\right)^2 + fK\Gamma^{\sqrt{\cdot}}(f), \quad f \in C^+(V).$$

By dividing by $f$ (all allowed $f \in C(V)$ are strictly positive), this is equivalent to

$$\Gamma_2^{\sqrt{\cdot}}(f) \geq \frac{1}{4d} \left(\Delta^{\log} f\right)^2 + K\Gamma^{\sqrt{\cdot}}(f), \quad f \in C^+(V).$$

By definition, this is equivalent to $CD_{\sqrt{\cdot}}^{\log}(4d, K)$. This finishes the proof. $\square$

## 3 The $CDE'$ inequality implies the $CD$ inequality

First, we recall a limit theorem [5, Theorem 3.18] by which it is shown that the $CD\psi$ condition implies the $CD$ condition (cf. [5, Corollary 3.20]).

**Theorem 3.1** (Limit of the $\psi$-operators)**.** *Let $G = (V, E)$ be a finite graph. Then for all $f \in C(V)$, one has the pointwise limits*

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Delta^{\psi}(1 + \varepsilon f) &= \psi'(1)\Delta f && \text{for } \psi \in C^1(\mathbb{R}^+), && (3.1)\\
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma^{\psi}(1 + \varepsilon f) &= -\psi''(1)\Gamma(f) && \text{for } \psi \in C^2(\mathbb{R}^+), && (3.2)\\
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma_2^{\psi}(1 + \varepsilon f) &= -\psi''(1)\Gamma_2(f) && \text{for } \psi \in C^2(\mathbb{R}^+). && (3.3)
\end{aligned}$$

*Since all $f \in C(V)$ are bounded, one obviously has $1 + \varepsilon f > 0$ for small enough $\varepsilon > 0$.*

*Proof.* For a proof, we refer the reader to the proof of [5, Theorem 3.18]. □

By adapting the methods of the proof of [5, Corollary 3.20], we can show that $CD_{\psi}^{\varphi}$ implies $CD$ and, especially, we can handle the $CDE'$ condition.

**Theorem 3.2.** *Let $\varphi, \psi \in C^2(\mathbb{R}^+)$ be concave with $\psi''(1) \neq 0 \neq \varphi'(1)$ and let $d \in \mathbb{R}^+$. Let $G = (V, E)$ be a graph satisfying the $CD_{\psi}^{\varphi}(d, K)$ condition. Then, $G$ also satisfies the $CD\left(\frac{-\psi''(1)}{\varphi'(1)^2} d, K\right)$ condition.*

*Proof.* Let $f \in C(V)$. We apply [5, Theorem 3.18] in the following both equations and since $G$ satisfies the $CD_{\psi}^{\varphi}(d, K)$ condition,

$$\begin{aligned}
-\psi''(1)\Gamma_2(f) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Gamma_2^{\psi}(1 + \varepsilon f) &\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left( \frac{1}{d} \left[\Delta^{\varphi}(1 + \varepsilon f)\right]^2 + K\Gamma^{\psi}(1 + \varepsilon f) \right)\\
&= \frac{\varphi'(1)^2}{d} (\Delta f)^2 - \psi''(1) K \Gamma(f).
\end{aligned}$$

Since $\psi$ is concave and $\psi''(1) \neq 0$, one has $-\psi''(1) > 0$. Thus, we obtain that $G$ satisfies the $CD\left(\frac{-\psi''(1)}{\varphi'(1)^2} d, 0\right)$ condition. □

**Corollary 3.3.** *If $G = (V, E)$ satisfies the $CDE'(d, K)$, i.e., the $CD_{\sqrt{\cdot}}^{\log}(4d, K)$, then $G$ also satisfies the $CD(d, K)$ condition since $-4\sqrt{\cdot}''(1) = 1 = \log'(1)$.*

## 4 The $CDE'$ inequality on Ricci-flat graphs

In [4], the $CDE'$ inequality is introduced. Examples for graphs satisfying this inequality have not been provided yet. In this section, we show that the more general $CD_{\psi}^{\varphi}$ condition holds on Ricci-flat graphs (cf. [3]). We will refer to the proof of the $CD\psi$ inequality on Ricci-flat graphs (cf. [5, Theorem 6.6]). Similarly to [5], we introduce a constant $C_{\psi}^{\varphi}$ describing the relation between the degree of the graph and the dimension parameter in the $CD_{\psi}^{\varphi}$ inequality.

**Definition 4.1.** Let $\varphi, \psi \in C^1(\mathbb{R})$. Then for all $x, y > 0$, we write

$$\widetilde{\psi}(x,y) := \left[\psi'(x) + \psi'(y)\right](1 - xy) + x[\psi(y) - \psi(1/x)] + y[\psi(x) - \psi(1/y)]$$

and

$$C_\psi^\varphi := \inf_{(x,y)\in A_\varphi} \frac{\widetilde{\psi}(x,y)}{(\varphi(x) + \varphi(y) - 2\varphi(1))^2} \in [-\infty, \infty]$$

with $A_\varphi := \{(x,y) \in (\mathbb{R}^+)^2 : \varphi(x) + \varphi(y) \neq 2\varphi(1)\}$. We have $C_\psi^\varphi = \infty$ iff $A_\varphi = \emptyset$.

**Theorem 4.2** ($CD_\psi^\varphi$ for Ricci-flat graphs)**.** *Let $D \in \mathbb{N}$, let $G = (V, E)$ be a $D$-Ricci-flat graph, and let $\psi, \varphi \in C^1(\mathbb{R}^+)$ be concave functions, such that $C_\psi^\varphi > 0$. Then, $G$ satisfies the $CD_\psi^\varphi(d, 0)$ inequality with $d = D/C_\psi^\varphi$.*

*Proof.* We can assume $\psi(1) = 0$ without loss of generality since $\Gamma_2^\psi$, $\Delta^\psi$ and $C_\psi$ are invariant under adding constants to $\psi$. Let $v \in V$ and $f \in C(V)$. Since $G$ is Ricci-flat, there are maps $\eta_1, \ldots, \eta_D : N(v) := \{v\} \cup \{w \sim v\} \to V$ as demanded in Definition A.5. For all $i, j \in \{1, \ldots, D\}$, we denote $y := f(v)$, $y_i := f(\eta_i(v))$, $y_{ij} := f(\eta_j(\eta_i(v)))$, $z_i := y_i/y$, $z_{ij} := y_{ij}/y_i$.

We take the sequence of inequalities at the end of the proof of [5, Theorem 6.6]. First, we extract the inequality

$$2\Gamma_2^\psi(f)(v) \geq \frac{1}{2}\sum_i \widetilde{\psi}(z_i, z_{i'}).$$

with $\psi$ and for an adequate permutation $i \mapsto i'$.

Secondly instead of continuing this estimate as in the proof of [5, Theorem 6.6], we take the latter part applied with $\varphi$ instead of $\psi$ to see

$$\frac{1}{2}\sum_i \left[\varphi(z_i) + \varphi(z_{i'})\right]^2 \geq \frac{2}{D}\left[\Delta^\varphi f(v)\right]^2.$$

Since $\widetilde{\psi}(z_i, z_{i'}) \geq C_\psi^\varphi \left[\varphi(z_i) + \varphi(z_{i'})\right]^2$, we conclude

$$2\Gamma_2^\psi(f)(v) \geq \frac{2C_\psi^\varphi}{D}\left[\Delta^\varphi f(v)\right]^2.$$

This finishes the proof. $\square$

The above theorem reduces the problem, whether $CD_\psi^\varphi$ holds on Ricci-flat graphs, to the question whether $C_\psi^\varphi > 0$. By using this fact, we can give the example of the $CDE'$ condition on Ricci-flat graphs.

**Example 4.3.** Numerical computations indicate that $C_{\sqrt{\cdot}}^{\log} > 0.1104$. Consequently by Theorem 4.2, $d$-Ricci-flat graphs satisfy the $CD_{\sqrt{\cdot}}^{\log}(9.058d, 0)$ inequality and thus due to Proposition 2.3, also the $CDE'(2.265d, 0)$ inequality.

Now, we give an analytic estimate of $C_{\sqrt{\cdot}}^{\log}$ by using methods similar to the proof of [5, Example 6.11] which shows $C_{\log}^{\log} \geq 1/2$.

**Lemma 4.4.** $C^{\log}_{\sqrt{\cdot}} \geq 1/16 = 0.0625$.

*Proof.* For $\psi = \sqrt{\cdot}$, we write

$$\begin{aligned}
\widetilde{\sqrt{\cdot}}(x,y) = \widetilde{\psi}(x,y) &= [\psi'(x) + \psi'(y)]\,(1-xy) + x[\psi(y) - \psi(1/x)] + y[\psi(x) - \psi(1/y)] \\
&= \left[\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\right](1-xy) + x\left[\sqrt{y} - \frac{1}{\sqrt{x}}\right] + y\left[\sqrt{x} - \frac{1}{\sqrt{y}}\right] \\
&= \frac{\sqrt{x}+\sqrt{y}}{2} \cdot \left(\frac{1}{\sqrt{xy}} - \sqrt{xy}\right) + \left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{xy}-1\right) \\
&= \frac{\sqrt{x}+\sqrt{y}}{2} \cdot \left((xy)^{1/4} - (xy)^{-1/4}\right)^2 \\
&\geq (xy)^{1/4} \cdot \left((xy)^{1/4} - (xy)^{-1/4}\right)^2 .
\end{aligned}$$

Hence by substituting $e^{2t} := (xy)^{1/4}$,

$$\begin{aligned}
\frac{\widetilde{\sqrt{\cdot}}(x,y)}{(\log x + \log y)^2} &\geq (xy)^{1/4} \cdot \left(\frac{(xy)^{1/4} - (xy)^{-1/4}}{4\log(xy)^{1/4}}\right)^2 \\
&= e^{2t} \cdot \left(\frac{e^{2t} - e^{-2t}}{8t}\right)^2 \\
&= \left(\frac{e^{3t} - e^{-t}}{8t}\right)^2 .
\end{aligned}$$

We expand the fraction to

$$\frac{e^{3t} - e^{-t}}{8t} = \frac{e^{3t} - e^{-t}}{e^{t} - e^{-t}} \cdot \frac{e^{t} - e^{-t}}{8t}.$$

Moreover,

$$\frac{e^{3t} - e^{-t}}{e^{t} - e^{-t}} = e^{2t} + 1 \geq 1$$

and, by the estimate $\frac{\sinh t}{t} \geq 1$,

$$\frac{e^{t} - e^{-t}}{8t} \geq 1/4.$$

Putting together the above estimates yields

$$C^{\log}_{\sqrt{\cdot}} = \inf_{x,y>0, xy\neq 1} \frac{\widetilde{\sqrt{\cdot}}(x,y)}{(\log x + \log y)^2} \geq (1/4)^2 = 1/16.$$

This finishes the proof. ∎

# A Appendix

**Definition A.1** (Graph)**.** A pair $G = (V, E)$ with a finite set $V$ and a relation $E \subset V \times V$ is called a *finite graph* if $(v, v) \notin E$ for all $v \in V$ and if $(v, w) \in E$ implies $(w, v) \in E$ for $v, w \in V$. For $v, w \in V$, we write $v \sim w$ if $(v, w) \in E$.

**Definition A.2** (Laplacian $\Delta$)**.** Let $G = (V, E)$ be a finite graph. The *Laplacian* $\Delta : C(V) := \mathbb{R}^V \to C(V)$ is defined for $f \in C(V)$ and $v \in V$ as $\Delta f(v) := \sum_{w \sim v}(f(w) - f(v))$.

**Definition A.3.** We write $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}_0^+ := [0, \infty)$. Let $G = (V, E)$ be a finite graph. Then, we write $C^+(V) := \{f : V \to \mathbb{R}^+\}$.

**Definition A.4** (Heat operator $\mathcal{L}$)**.** Let $G = (V, E)$ be a graph. The *heat operator* $\mathcal{L} : C^1(V \times \mathbb{R}^+) \to C(V \times \mathbb{R}^+)$ is defined by $\mathcal{L}(u) := \Delta u - \partial_t u$ for all $u \in C^1(V \times \mathbb{R}^+)$. We call a function $u \in C^1(V \times \mathbb{R}_0^+)$ a *solution to the heat equation* on $G$ if $\mathcal{L}(u) = 0$.

**Definition A.5** (Ricci-flat graphs)**.** Let $D \in \mathbb{N}$. A finite graph $G = (V, E)$ is called *$D$-Ricci-flat* in $v \in V$ if all $w \in N(v) := \{v\} \cup \{w \in V : w \sim v\}$ have the degree $D$, and if there are maps $\eta_1, \dots, \eta_D : N(v) \to V$, such that for all $w \in N(v)$ and all $i, j \in \{1, \dots, D\}$ with $i \neq j$, one has $\eta_i(w) \sim w$, $\eta_i(w) \neq \eta_j(w)$, $\bigcup_k \eta_k(\eta_i(v)) = \bigcup_k \eta_i(\eta_k(v))$. The graph $G$ is called *$D$-Ricci-flat* if it is $D$-Ricci-flat in all $v \in V$.

## A.1 The $CD$ condition via $\Gamma$ calculus

We give the definition of the $\Gamma$-calculus and the $CD$ condition following [1].

**Definition A.6** ($\Gamma$-calculus)**.** Let $G = (V, E)$ be a finite graph. Then, the *gradient form* or *carré du champ* operator $\Gamma : C(V) \times C(V) \to C(V)$ is defined by

$$2\Gamma(f, g) := \Delta(fg) - f\Delta g - g\Delta f.$$

Similarly, the *second gradient form* $\Gamma_2 : C(V) \times C(V) \to C(V)$ is defined by

$$2\Gamma_2(f, g) := \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f).$$

We write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f) := \Gamma_2(f, f)$.

**Definition A.7** ($CD(d, K)$ condition)**.** Let $G = (V, E)$ be a finite graph and $d \in \mathbb{R}^+$. We say $G$ satisfies the *curvature-dimension inequality* $CD(d, K)$ if for all $f \in C(V)$,

$$\Gamma_2(f) \geq \frac{1}{d}(\Delta f)^2 + K\Gamma(f).$$

We can interpret this as meaning that that the graph $G$ has a dimension (at most) $d$ and a Ricci curvature larger than $K$.

## A.2 The $CDE$ and $CDE'$ conditions via $\widetilde{\Gamma_2}$

We give the definitions of $CDE$ and $CDE'$ following [2, 4]

**Definition A.8** (The $CDE$ inequality)**.** We say that a graph $G = (V, E)$ satisfies the $CDE(x, d, K)$ inequality if for any $f \in C^+(V)$ such that $\Delta f(x) < 0$, we have

$$\widetilde{\Gamma_2}(f)(x) := \Gamma_2(f)(x) - \Gamma\left(f, \frac{\Gamma(f)}{f}\right)(x) \geq \frac{1}{d}\left(\Delta f\right)^2(x) + K\Gamma(f)(x).$$

We say that $CDE(d, k)$ is satisfied if $CDE(x, d, K)$ is satisfied for all $x \in V$.

**Definition A.9** (The $CDE'$ inequality)**.** We say that a graph $G = (V, E)$ satisfies the $CDE'(d, K)$ inequality if for any $f \in C^+(V)$, we have

$$\widetilde{\Gamma_2}(f) \geq \frac{1}{d} f^2 \left(\Delta \log f\right)^2 + K\Gamma(f).$$

### A.3 The $CD\psi$ conditions via $\Gamma^\psi$ calculus

We give the definition of the $\Gamma^\psi$-calculus and the $CD\psi$ condition following [5].

**Definition A.10** ($\psi$-Laplacian $\Delta^\psi$)**.** Let $\psi \in C^1(\mathbb{R}^+)$ and let $G = (V, E)$ be a finite graph. Then, we call $\Delta^\psi : C^+(V) \to C(V)$, defined as

$$(\Delta^\psi f)(v) := \left(\Delta \left[\psi\left(\frac{f}{f(v)}\right)\right]\right)(v),$$

the *$\psi$-Laplacian.*

**Definition A.11** ($\psi$-gradient $\Gamma^\psi$)**.** Let $\psi \in C^1(\mathbb{R}^+)$ be a concave function and let $G = (V, E)$ be a finite graph. We define

$$\overline{\psi}(x) := \psi'(1) \cdot (x - 1) - (\psi(x) - \psi(1)).$$

Moreover, we define the *$\psi$-gradient* as $\Gamma^\psi : C^+(V) \to C(V)$,

$$\Gamma^\psi := \Delta^{\overline{\psi}}.$$

**Definition A.12** (Second $\psi$-gradient $\Gamma_2^\psi$)**.** Let $\psi \in C^1(\mathbb{R}^+)$, and let $G = (V, E)$ be a finite graph. Then, we define $\Omega^\psi : C^+(V) \to C(V)$ by

$$(\Omega^\psi f)(v) := \left(\Delta \left[\psi'\left(\frac{f}{f(v)}\right) \cdot \frac{f}{f(v)} \left[\frac{\Delta f}{f} - \frac{(\Delta f)(v)}{f(v)}\right]\right]\right)(v).$$

Furthermore, we define the *second $\psi$-gradient* $\Gamma_2^\psi : C^+(V) \to C(V)$ by

$$2\Gamma_2^\psi(f) := \Omega^\psi f + \frac{\Delta f \Delta^\psi f}{f} - \frac{\Delta\left(f \Delta^\psi f\right)}{f}.$$

**Definition A.13** ($CD\psi$ condition)**.** Let $G = (V, E)$ be a finite graph, $K \in \mathbb{R}$ and $d \in \mathbb{R}^+$. We say $G$ satisfies the $CD\psi(d, K)$ *inequality* if for all $f \in C^+(V)$, one has

$$\Gamma_2^\psi(f) \geq \frac{1}{d}\left(\Delta^\psi f\right)^2 + K\Gamma^\psi(f).$$

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